

ON THE BRAUER-MANIN OBSTRUCTION FOR DEGREE FOUR DEL PEZZO SURFACES

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ABSTRACT. We show that, for every integer $1 \leq d \leq 4$ and every finite set S of places, there exists a degree d del Pezzo surface X over \mathbb{Q} such that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the Brauer-Manin obstruction works exactly at the places in S . For $d = 4$, we prove that in all cases, with the exception of $S = \{\infty\}$, this surface may be chosen diagonalizably over \mathbb{Q} .

1. INTRODUCTION

A del Pezzo surface is a smooth, proper algebraic surface X over a field K with an ample anti-canonical sheaf \mathcal{K}^{-1} . Over an algebraically closed field, every del Pezzo surface of degree $d \leq 7$ is isomorphic to \mathbf{P}^2 , blown up in $(9 - d)$ points in general position [Man, Theorem 24.4.iii].

According to the adjunction formula, a smooth complete intersection of two quadrics in \mathbf{P}^4 is del Pezzo. The converse is true, as well. For every del Pezzo surface of degree four, its anticanonical image is the complete intersection of two quadrics in \mathbf{P}^4 [Do, Theorem 8.6.2].

For an arbitrary proper variety X over \mathbb{Q} , the Brauer-Manin obstruction is a phenomenon that can explain failures of weak approximation or even the Hasse principle. Its mechanism works as follows.

Let p be any prime number. The Grothendieck-Brauer group is a contravariant functor from the category of schemes to the category of abelian groups. In particular, for an arbitrary scheme X and a \mathbb{Q}_p -rational point $x: \mathrm{Spec} \mathbb{Q}_p \rightarrow X$, there is a restriction homomorphism $x^*: \mathrm{Br}(X) \rightarrow \mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$. For a Brauer class $\alpha \in \mathrm{Br}(X)$, we call

$$\mathrm{ev}_{\alpha,p}: X(\mathbb{Q}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad x \mapsto x^*(\alpha),$$

the local evaluation map, associated to α . Analogously, for the real place, there is the local evaluation map $\mathrm{ev}_{\alpha,\infty}: X(\mathbb{R}) \rightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Let us use the notation Ω for the set of all places of \mathbb{Q} , i.e. for the union of all finite primes together with ∞ . The local evaluation maps are continuous with respect to the p -adic, respectively real, topologies on $X(\mathbb{Q}_\nu)$. Moreover, it is well-known that $\mathrm{ev}_{\alpha,\nu}$ is constant for all but finitely many places. Thus, only adelic

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points $x = (x_\nu)_{\nu \in \Omega} \in X(\mathbb{A}_{\mathbb{Q}})$ satisfying

$$(1) \quad \sum_{\nu \in \Omega} \text{ev}_{\alpha, \nu}(x_\nu) = 0 \in \mathbb{Q}/\mathbb{Z}$$

may possibly be approximated by \mathbb{Q} -rational points.

We say that the Brauer class $\alpha \in \text{Br}(X)$ *works* at a place ν if the local evaluation map $\text{ev}_{\alpha, \nu}: X(\mathbb{Q}_\nu) \rightarrow \mathbb{Q}/\mathbb{Z}$ is non-constant. This is in fact a property of the residue class of α in $\text{Br}(X)/\text{Br}(\mathbb{Q})$.

Observe that if $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ and there exists a Brauer class that works at least at a single place then weak approximation is violated on X . On the other hand, if $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and a generator works at least at one place then there are adelic points fulfilling (1). That is, the Brauer-Manin obstruction cannot explain a violation of the Hasse principle.

The goal of this paper is to investigate which subsets of Ω may occur as the set of places, at which a nontrivial Brauer class works, in the situation of a degree four del Pezzo surface. Our first main result is as follows.

Theorem 1.1. *Let $S \subset \Omega$ be any finite subset. Then there exists a degree four del Pezzo surface X over \mathbb{Q} having a \mathbb{Q} -rational point such that $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial Brauer class works exactly at the places in S .*

In particular, there is the following example.

Example 1.2. Let $X \subset \mathbf{P}_{\mathbb{Q}}^4$ be the degree four del Pezzo surface that is given by the equations

$$\begin{aligned} T_0 T_1 &= T_2^2 + 7T_3^2, \\ (T_0 - 4T_1)(T_0 - 6T_1) &= T_2^2 + 7T_4^2. \end{aligned}$$

Then X has a \mathbb{Q} -rational point, $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$, and the nontrivial Brauer class works exactly at the infinite place. In particular, the surface $X(\mathbb{R})$ has two connected components and \mathbb{Q} -rational points on only one of them.

More details about this example are given in Remark 2.7. The particular case that $S = \{\infty\}$ is perhaps the most interesting one. Indeed, there is a relation to Mazur's conjecture [Maz, Conjecture 1] stating that the closure of $X(\mathbb{Q})$ with respect to the real topology is equal to a union of connected components. Similar examples for other kinds of surfaces are available in the literature, including singular cubic surfaces [SD1, §3], conic bundles with five singular fibers [Maz, §3], and others.

Recall that over an algebraically closed field two quadratic forms are always simultaneously diagonalizable. We say that a degree four del Pezzo surface is diagonalizable over \mathbb{Q} if the defining quadratic forms are diagonalizable over \mathbb{Q} .

The surface from Example 1.2 is not diagonalizable over \mathbb{Q} but only over $\mathbb{Q}(\sqrt{6})$, as is easily seen using Fact 2.1.b.iii). Somewhat surprisingly, such a behaviour is necessary at this point. There is the following result.

Theorem 1.3. *Let X be a degree four del Pezzo surface over \mathbb{Q} having an adelic point and $\alpha \in \text{Br}(X)$ a Brauer class that works exactly at the infinite place. Then X is not diagonalizable over \mathbb{Q} .*

Our method of proof uses the fact that diagonal degree four del Pezzo surfaces have nontrivial automorphisms. By functoriality, these operate on $\text{Br}(X)$, but the induced operation on $\text{Br}(X)/\text{Br}(\mathbb{Q})$ turns out to be trivial automatically. Therefore, every $\alpha \in \text{Br}(X)/\text{Br}(\mathbb{Q})$ induces a homomorphism $i_\alpha: \text{Aut}'(X) \rightarrow \text{Br}(K)$. Cf. Construction 3.2 for more details.

Moreover, we prove that if $\alpha \in \text{Br}(X)$ works at ∞ then there is an automorphism $\sigma \in \text{Aut}(X)$ witnessing this, i.e. such that $i_\alpha(\sigma)$ has a nontrivial component at ∞ . From this, the claim easily follows.

Our third main result asserts that, for diagonalizable degree four del Pezzo surfaces, the subset $\{\infty\}$ is the only exception of this kind.

Theorem 1.4. *Let $S \subset \Omega$ be a finite subset, different from $\{\infty\}$. Then there exists a diagonalizable degree four del Pezzo surface X over \mathbb{Q} having a \mathbb{Q} -rational point such that $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial Brauer class works exactly at the places in S .*

Conjecturally, for degree four del Pezzo surfaces, all failures of weak approximation are due to the Brauer-Manin obstruction. More precisely, it is conjectured that $X(\mathbb{Q})$ is dense in

$$X(\mathbb{A}_{\mathbb{Q}})^{\text{Br}} := \bigcap_{\alpha \in \text{Br}(X)} X(\mathbb{A}_{\mathbb{Q}})^{\alpha},$$

for $X(\mathbb{A}_{\mathbb{Q}})^{\alpha} \subseteq X(\mathbb{A}_{\mathbb{Q}})$ the subset defined by condition (1) and $X(\mathbb{A}_{\mathbb{Q}})$ endowed with the product topology induced by the ν -adic topologies on $X(\mathbb{Q}_{\nu})$.

Due to work of P. Salberger and A. N. Skorobogatov [SSk, Theorem 0.1], this conjecture is proven under the assumption that X has a \mathbb{Q} -rational point. In particular, if X has a \mathbb{Q} -rational point then the \mathbb{Q} -rational points on X are automatically Zariski dense.

Recall that all the surfaces provided by Theorem 1.1 have a \mathbb{Q} -rational point. We may thus blow up \mathbb{Q} -rational points in general position to obtain del Pezzo surfaces of low degree, thereby unconditionally establishing the following result.

Theorem 1.5. *Let $S \subset \Omega$ be an arbitrary finite subset and $d \leq 4$ a positive integer. Then there exists a del Pezzo surface X of degree d over \mathbb{Q} having a \mathbb{Q} -rational point such that $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial Brauer class works exactly at the places in S .*

It is well-known that every del Pezzo surface X of degree at least five has $\text{Br}(X)/\text{Br}(\mathbb{Q}) = 0$. One way to see this is to systematically inspect all possible Galois operations on the exceptional curves in a way analogous to [Ja, Chapter III, 8.21–8.23] and to apply [Man, Proposition 31.3]. Thus, Theorem 1.5 cannot have an analogue for del Pezzo surfaces of higher degree.

At least for $d = 5$ and 7 , as well as for $d = 6$ under the additional assumption that X has an adelic point, there is also a geometric argument. Indeed, these surfaces are birationally equivalent to $\mathbf{P}_{\mathbb{Q}}^2$ [VA, Theorem 2.1], cf. [Man, Theorem 29.4].

Remark 1.6. For a degree four del Pezzo surface, the group $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ may be isomorphic to either 0 , $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$. In the cases of degree 3, 2, or 1, there are even more options [Man, Section 31, Table 3] and [SD2]. We do not know whether the analogue of Theorem 1.5 is true for a prescribed Brauer group.

2. BRAUER CLASSES ON DEGREE FOUR DEL PEZZO SURFACES

The goal of this section is to gather some facts about degree four del Pezzo surfaces that are necessary for the following. This includes some results on their Brauer groups and finally leads us to a proof of the assertions made in Example 1.2. In other words, we show Theorem 1.1 under the assumption of Theorem 1.4. Unless a specific choice is made, we work in this section over an arbitrary base field K of characteristic $\neq 2$. Let us denote by \overline{K} an algebraic closure of K .

A del Pezzo surface $X \subset \mathbf{P}_K^4$ of degree four is the base locus of a pencil $(\mu Q^{(1)} + \nu Q^{(2)})_{(\mu, \nu) \in \mathbf{P}^1}$ of quadratic forms in five variables with coefficients in the field K . The generic member of the pencil must be of rank five, as otherwise X would be a cone. The condition that $\det(\mu Q^{(1)} + \nu Q^{(2)}) = 0$ therefore defines a finite subscheme $\mathcal{S}_X \subset \mathbf{P}_K^1$ of degree 5.

Choosing a different basis of the pencil yields another embedding of \mathcal{S}_X into the projective line. Thus, one may consider the subscheme $\mathcal{S}_X \subset \mathbf{P}_K^1$ as an invariant of the surface X itself. Moreover, the definition extends to arbitrary intersections of two quadrics in \mathbf{P}^4 that are not cones.

Facts 2.1. a) X is nonsingular if and only if the scheme \mathcal{S}_X is reduced.

b) Let $X \subset \mathbf{P}^4$ be a smooth intersection of two quadrics. Then the following statements hold.

i) If $\{s_0, \dots, s_4\} = \mathcal{S}_X(\overline{K})$ then the quadratic forms Q_{s_0}, \dots, Q_{s_4} are exactly of rank 4.

ii) The cusps of the cones defined by $Q_{s_i} = 0$, for $i = 0, \dots, 4$, are in general linear position in \mathbf{P}^4 , i.e. not contained in any hyperplane.

iii) X is diagonalizable over K if and only if \mathcal{S}_X is split over K .

Proof. These statements are rather well-known. Proofs may be found, for example, in [Wi]. Concretely, parts a) and b.i) are implied by [Wi, Proposition 3.26]. Furthermore, part b.ii) is [Wi, Corollaire 3.29], while part b.iii) is [Wi, Corollaire 3.30]. \square

Let X be a degree four del Pezzo surface over a field K and assume that there is a K -rational point $s \in \mathcal{S}_X(K)$ as well as that the corresponding degenerate quadric Q_s has a K -rational point, different from the cusp. Then there exist four linearly independent linear forms l_1, \dots, l_4 such that

$$Q_s = l_1 l_2 - (l_3^2 - D l_4^2).$$

Furthermore, D is the discriminant of Q_s , considered as a quadratic form in four variables.

Indeed, Q_s represents zero nontrivially and it is well-known that such a quadratic form splits off a hyperbolic plane [Se, Chapitre 4, Proposition 3]. More geometrically, one may argue as follows. Take l_1 to be a linear form that describes the hyperplane tangent to the cone defined by Q_s at a nonsingular K -rational point. The restriction of Q_s to this hyperplane is of rank two. After scaling, it may be written in the form $l_3^2 - Dl_4^2$. Finally, $Q_s - (l_3^2 - Dl_4^2)$ is a quadratic form that vanishes on the hyperplane defined by l_1 and therefore splits.

The four linear forms l_1, \dots, l_4 must be linearly independent as Q_s is of rank four. Hence, the quadratic form Q_s is equivalent to $T_0T_1 - (T_2^2 - DT_3^2)$, which has discriminant D .

The case most interesting for us is when there are two distinct K -rational points $s_1, s_2 \in \mathcal{S}_X(K)$ and the corresponding degenerate quadrics Q_{s_1}, Q_{s_2} have the same discriminant. Then X may be given by a system of equations in the form

$$(2) \quad l_{11}l_{12} = l_{13}^2 - Dl_{14}^2,$$

$$(3) \quad l_{21}l_{22} = l_{23}^2 - Dl_{24}^2.$$

For such surfaces, there is a standard way to write down a Brauer class, which goes back, at least to B. Birch and Sir Peter Swinnerton-Dyer [BSD].

Proposition 2.2. *Let X be the degree four del Pezzo surface over a field K , given by the equations (2,3). Assume that D is a non-square in K and put $L := K(\sqrt{D})$.*

a) *Then the quaternion algebra (see [Pi, Section 15.1] for the notation)*

$$\mathcal{A} := (L(X), \tau, \frac{l_{11}}{l_{21}})$$

over the function field $K(X)$ extends to an Azumaya algebra over the whole of X . Here, by $\tau \in \text{Gal}(L(X)/K(X))$, we denote the nontrivial element.

b) *In the case that $K = \mathbb{Q}$, denote by $\alpha \in \text{Br}(X)$ the Brauer class, defined by the extension of \mathcal{A} . Let ν be any (archimedean or non-archimedean) place of \mathbb{Q} .*

i) *Let $x \in X(\mathbb{Q}_\nu)$ be a point and assume that, for some $i, j \in \{1, 2\}$, one has $l_{1i}(x) \neq 0$ and $l_{2j}(x) \neq 0$. Denote the corresponding quotient $l_{1i}(x)/l_{2j}(x)$ by q . Then*

$$\text{ev}_{\alpha, \nu}(x) = \begin{cases} 0 & \text{if } (q, D)_\nu = 1, \\ \frac{1}{2} & \text{if } (q, D)_\nu = -1, \end{cases}$$

for $(q, D)_\nu$ the Hilbert symbol.

ii) *If ν is split in L then the local evaluation map $\text{ev}_{\alpha, \nu}$ is constantly zero.*

Proof. a) First of all, \mathcal{A} is, by construction, a cyclic algebra of degree two. In particular, \mathcal{A} is simple [Pi, Section 15.1, Corollary d]. Moreover, \mathcal{A} is obviously a central $K(X)$ -algebra.

To prove the extendability assertion, it suffices to show that \mathcal{A} extends as an Azumaya algebra over each valuation ring that corresponds to a prime divisor on X .

Indeed, this is the classical Theorem of Auslander-Goldman for non-singular surfaces [AG, Proposition 7.4], cf. [Mi, Chapter IV, Theorem 2.16].

To verify this, we observe that the principal divisor $\text{div}(l_{11}/l_{21}) \in \text{Div}(X)$ is the norm of a divisor on X_L . In fact, it is the norm of the difference of two prime divisors, the conic, given by $l_{11} = l_{13} - \sqrt{D}l_{14} = 0$, and the conic, given by $l_{21} = l_{23} - \sqrt{D}l_{24} = 0$. In particular, \mathcal{A} defines the zero element in $H^2(\langle\sigma\rangle, \text{Div}(X_L))$. Under such circumstances, the extendability of \mathcal{A} over the valuation ring corresponding to an arbitrary prime divisor on X is worked out in [Man, Paragraph 42.2].

b.i) The quotients

$$\frac{l_{11}}{l_{21}} / \frac{l_{11}}{l_{22}} = \frac{l_{23}^2 - D l_{24}^2}{l_{21}^2}, \quad \frac{l_{12}}{l_{21}} / \frac{l_{12}}{l_{22}} = \frac{l_{23}^2 - D l_{24}^2}{l_{21}^2}, \quad \text{and} \quad \frac{l_{11}}{l_{21}} / \frac{l_{12}}{l_{21}} = \frac{l_{13}^2 - D l_{14}^2}{l_{12}^2}$$

are norms of rational functions from $L(X)$. Therefore, they define the trivial element of $H^2(\langle\sigma\rangle, K(X_L)^*) \subseteq \text{Br } K(X)$, and hence in $\text{Br } X$. In particular, the four expressions l_{1i}/l_{2j} define the same Brauer class.

The general description of the evaluation map, given in [Man, Paragraph 45.2], shows that $\text{ev}_{\alpha, \nu}(x)$ is equal to 0 or $\frac{1}{2}$ depending on whether q is in the image of the norm map $N_{L_n/\mathbb{Q}_\nu}: L_n^* \rightarrow \mathbb{Q}_\nu^*$, or not, for \mathfrak{n} a place of L lying above ν . This is exactly what is tested by the Hilbert symbol $(q, D)_\nu$.

ii) If ν is split in L then the norm map $N_{K(X_{L_n})/K(X_{\mathbb{Q}_\nu})}: K(X_{L_n})^* \rightarrow K(X_{\mathbb{Q}_\nu})^*$ is surjective. In particular, $l_{11}/l_{21} \in K(X_{\mathbb{Q}_\nu})^*$ is the norm of a rational function on X_{L_n} . Therefore, it defines the zero class in $H^2(\langle\sigma\rangle, K(X_{L_n})^*) \subseteq \text{Br } K(X_{\mathbb{Q}_\nu})$, and thus in $\text{Br } X_{\mathbb{Q}_\nu}$. To complete the argument, we note that every \mathbb{Q}_ν -rational point $x: \text{Spec } \mathbb{Q}_\nu \rightarrow X$ factors via $X_{\mathbb{Q}_\nu}$. \square

In the following, we will make heavy use of the two facts below. The first one recalls the explicit description of the situation when the Brauer group of X is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Fact 2.3. *Let X be a degree four del Pezzo surface over a local or global field K . In the local field case, assume that $X(K) \neq \emptyset$, and in the global field case that X has an adelic point.*

Then $\text{Br}(X)/\text{Br}(K) \cong (\mathbb{Z}/2\mathbb{Z})^2$ if and only if \mathcal{S}_X has three distinct points $s_0, s_1, s_2 \in \mathcal{S}_X(K)$ so that all three discriminants $D_{s_0}, D_{s_1}, D_{s_2}$ are non-squares in K and coincide up to square factors.

In this case, representatives of the three nontrivial classes may be obtained as follows. Choose a subset $\{s_i, s_j\} \subset \{s_0, s_1, s_2\}$ of size two. Write X accordingly in the form $(2, 3)$ and take the corresponding Azumaya algebra as described in Proposition 2.2.

Proof. This is a well-known fact and a proof may be found, for example, in [VAV, Theorem 3.4]. Note that the assumption on X implies that, for every closed point $s \in \mathcal{S}_X$, the corresponding rank-4 quadric has a regular point over the residue field of s [VAV, Lemma 5.1]. \square

Fact 2.4. *Let X be a degree four del Pezzo surface over a local or global field K . In the local field case, suppose that $X(K) \neq \emptyset$, and in the global field case that X has an adelic point.*

Assume X to be diagonalizable over K . Let $D_i \in K$ for $0 \leq i \leq 4$ be the five rank-4 discriminants and assume that $D_0 = D_1 =: D$.

a) *Let D be a non-square in K . Then the Brauer class $\alpha \in \text{Br}(X)$ described in Proposition 2.2 is trivial, i.e. $\alpha \in \text{Br}(K)$, if and only if D_2, D_3 , and D_4 are all squares in K .*

b) *If the conditions in a) hold or all five discriminants D_i are squares in K , then one has $\text{Br}(X)/\text{Br}(K) \cong 0$.*

Proof. a) This equivalence statement is established in [VAV, Proposition 3.3].

b) Fact 2.3 above proves that $\text{Br}(X)/\text{Br}(K)$ is at most of order two. If it were of order exactly two then, by [VAV, Theorem 3.4], the nontrivial class could be obtained as described in Proposition 2.2. In particular, only the case that D is a non-square remains to be considered. However, as the other three discriminants are squares, this is exactly the situation in which part a) proves that the Brauer class is trivial. \square

Remark 2.5. Under the assumptions of Fact 2.4, there is an isomorphism

$$\text{Br}(X)/\text{Br}(K) \xleftarrow{\cong} \ker(o: (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow K^*/(K^*)^2)/T,$$

where the homomorphism o is given by $o: (a_0, \dots, a_4) \mapsto (D_0^{a_0} \dots D_4^{a_4} \bmod (K^*)^2)$ and T is generated by the vector $(1, \dots, 1)$ and the standard vectors e_i , for those $i \in \{0, \dots, 4\}$ for which D_i is a perfect square. Note that $D_0 \dots D_4$ is a perfect square in K , cf. Fact 3.4.

If X has a K -rational point x not lying on any exceptional curve then this may be seen roughly as follows. The blow-up $\text{Bl}_x(X)$ is a cubic surface with a K -rational line E . The planes through E equip $\text{Bl}_x(X)$ with a structure of a conic bundle. One can show that the five degenerate conics split exactly over $K(\sqrt{D_0}), \dots, K(\sqrt{D_4})$. The result then follows from the same argument as in [Sk, Proposition 7.1.1], cf. [BMS, remarks after Theorem 1.1].

It requires, however, quite a lot more effort to establish not only an abstract isomorphism, but to prove that the Brauer classes obtained are exactly those expected in view of Proposition 2.2.

Once one has an explicit description of the Brauer classes, one needs criteria to understand whether or not they evaluate constantly at a given place. For this the following result turns out to be very useful.

Criterion 2.6 (A. Várilly-Alvarado and B. Viray). *Let X be the degree four del Pezzo surface over \mathbb{Q} given by the equations (2.3). Assume that D is a non-square and let $\alpha \in \text{Br}(X)$ be the Brauer class described in Proposition 2.2.*

Then, for any place $\nu \neq 2, \infty$ such that the reductions modulo ν of the quadratic forms in (2) and (3) both have rank 4, the local evaluation map $\text{ev}_{\alpha, \nu}$ is constant.

Proof. This is [VAV, Proposition 5.2]. \square

Proof of Theorem 1.1 assuming Theorem 1.4. Theorem 1.4 solves the problem for every subset $S \neq \{\infty\}$. Thus, in order to establish Theorem 1.1, it suffices to verify the assertions made in Example 1.2.

For this, one first checks that \mathcal{S}_X has exactly three \mathbb{Q} -rational points, corresponding to the quadratic forms independent of the variable T_2 , T_3 , and T_4 , respectively, and a point of degree two that splits over the quadratic field $\mathbb{Q}(\sqrt{6})$. In particular, X is nonsingular.

The discriminants of the three \mathbb{Q} -rational quadratic forms of rank 4 are, up to square factors, 1, (-7) , and (-7) . Therefore, Fact 2.3 shows that $\text{Br}(X)/\text{Br}(\mathbb{Q})$ is at most of order 2. On the other hand, by Proposition 2.2, we have a Brauer class $\alpha \in \text{Br}(X)$ that is given over the function field $\mathbb{Q}(X)$ as the quaternion algebra $(\mathbb{Q}(\sqrt{-7})(X), \tau, \varphi)$ for $\varphi := \frac{T_0 - 4T_1}{T_1}$.

Next, we observe that X has no real points with $x_0 = x_1 = 0$. Moreover, for an arbitrary real point $x \in X(\mathbb{R})$ such that $x_1 \neq 0$, the equations imply $x_0/x_1 \geq 0$ and $(\frac{x_0}{x_1} - 4)(\frac{x_0}{x_1} - 6) \geq 0$, hence

$$x_0/x_1 \in [0, 4] \quad \text{or} \quad x_0/x_1 \geq 6.$$

There exist real points of both kinds, for example $(1:1:1:0:\sqrt{2})$ and $(8:1:1:1:1)$. Since $(-7) < 0$, we have that $(q, -7)_\infty$ is the sign of q . Thus, $\text{ev}_{\alpha, \infty}$ distinguishes the two kinds of real points. In particular, $\text{Br}(X)/\text{Br}(\mathbb{Q})$ is indeed of order two and the nontrivial element works at the infinite place.

It remains to show that it does not work at any other place. Criterion 2.6 shows constancy of the evaluation map $\text{ev}_{\alpha, \nu}$ for all finite places $\nu \neq 2, 7$. Furthermore, $\text{ev}_{\alpha, 2}$ is constant by Proposition 2.2.b.ii), as the prime 2 splits in $\mathbb{Q}(\sqrt{-7})$.

Finally, for the prime 7, we argue as follows. Let $x \in X(\mathbb{Q}_7)$ be any 7-adic point on X . Normalize the coordinates x_0, \dots, x_4 such that each is a 7-adic integer and at least one is a unit. If $7|x_0$ and $7|x_1$ then the equations imply that all coordinates must be divisible by 7, a contradiction. Hence, at least one of x_0 and x_1 is a unit. Modulo 7, we have $(\bar{x}_0 - 4\bar{x}_1)(\bar{x}_0 - 6\bar{x}_1) = \bar{x}_0\bar{x}_1$ (since both expressions are equal to \bar{x}_2^2), and this equation has the solutions $\bar{x}_0/\bar{x}_1 = 1, 3$ in $\mathbb{Z}/7\mathbb{Z}$. However, the solution $\bar{x}_0/\bar{x}_1 = 3$ is contradictory, as then $\bar{x}_0\bar{x}_1$ would be a non-square. Consequently, both x_0 and x_1 must be units and

$$\frac{x_0 - 4x_1}{x_1} \equiv -3 \pmod{7},$$

which implies that $\frac{x_0 - 4x_1}{x_1}$ is a square in \mathbb{Q}_7 . This shows $(\frac{x_0 - 4x_1}{x_1}, -7)_7 = 1$ and $\text{ev}_{\alpha, 7}(x) = 0$. \square

Remark 2.7. In Example 1.2, weak approximation is disturbed in a rather astonishing way. The smooth manifold $X(\mathbb{R})$ is disconnected into two components. There are two kinds of real points $x \in X(\mathbb{R})$, those with $x_0/x_1 \in [0, 4]$ and those

such that $x_0/x_1 \in [6, \infty]$. However, for every \mathbb{Q} -rational point $x \in X(\mathbb{Q})$, one has $x_0/x_1 > 6$.

A naively implemented point search shows that there are exactly 792 \mathbb{Q} -rational points of naive height up to 1000 on X . The smallest value of the quotient x_0/x_1 is $319/53 \approx 6.019$.

3. DIAGONAL DEGREE FOUR DEL PEZZO SURFACES

The goal of this section is to collect some facts about diagonal degree four del Pezzo surfaces. These will lead us to a proof of Theorem 1.3.

Let X be a diagonal degree four del Pezzo surface over a base field K , i.e. one that is given by equations of the form

$$(4) \quad a_0 T_0^2 + \dots + a_4 T_4^2 = 0,$$

$$(5) \quad b_0 T_0^2 + \dots + b_4 T_4^2 = 0$$

with coefficients in K . Then, for every $(i_0, \dots, i_4) \in \{0, 1\}^5$, the map

$$(T_0 : \dots : T_4) \mapsto ((-1)^{i_0} T_0 : \dots : (-1)^{i_4} T_4)$$

defines a K -automorphism of X . Thus, there is a subgroup $\text{Aut}'(X) \subseteq \text{Aut}_K(X)$ that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.

It is known that the automorphism group of a degree four del Pezzo surface over an algebraically closed field is generically isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ and that there are particular cases, where the automorphism group is larger [Do, Theorem 8.6.8].

Lemma 3.1. *Let X be a diagonal degree four del Pezzo surface over a local or global field K . In the local field case, suppose that $X(K) \neq \emptyset$, and in the global field case that X has an adelic point.*

Then the natural operation of $\text{Aut}'(X)$ on $\text{Br}(X)$ induces the trivial operation on $\text{Br}(X)/\text{Br}(K)$.

Proof. This is trivially true if $\text{Br}(X)/\text{Br}(K) \cong 0$ or $\mathbb{Z}/2\mathbb{Z}$. Otherwise, it follows from the description of the representatives given in Fact 2.3. \square

Construction 3.2. Let X be a diagonal degree four del Pezzo surface over a local or global field K . In the local field case, suppose that $X(K) \neq \emptyset$, and in the global field case that X has an adelic point.

By functoriality, the operation of $\text{Aut}'(X)$ on X induces an operation on $\text{Br}(X)$, which is necessarily trivial on $\text{Br}(X)/\text{Br}(K)$. Thus, for every $\alpha \in \text{Br}(X)/\text{Br}(K)$, there is a natural homomorphism

$$i_\alpha: \text{Aut}'(X) \longrightarrow \text{Br}(K),$$

given by the condition that $\sigma^* \alpha = \alpha + i(\sigma)$ for $\sigma \in \text{Aut}'(X)$.

Definition 3.3. Let $K = \mathbb{Q}$ and assume that a Brauer class $i_\alpha(\sigma)$ in the image of i_α has a nontrivial component at the place ν . Then, as

$$\text{ev}_{\alpha, \nu}(\sigma(x)) = \text{ev}_{\alpha, \nu}(x) + i(\sigma)_\nu,$$

the Brauer class certainly works at ν . We say in this situation that σ is a *witness* for the non-constancy of the local evaluation map at ν .

Fact 3.4. *Let X be a diagonal degree four del Pezzo surface over a field K and D_0, \dots, D_4 be the discriminants of the five associated quadratic forms of rank 4. Then $D_0 \cdots D_4$ is a square in K .*

Proof. This is a direct calculation. \square

Lemma 3.5. *Let X be a diagonal degree four del Pezzo surface over \mathbb{R} that has a real point. Assume that $\text{Br}(X)/\text{Br}(\mathbb{R}) \neq 0$.*

Then $X(\mathbb{R})$ splits into two connected components. Moreover, there is an element $\sigma \in \text{Aut}'(X)$ that interchanges these components.

Proof. By Fact 3.4, there are three cases. Either, no of the three rank-4 discriminants is negative, or exactly two, or exactly four of them. Fact 2.4.b) shows that $\text{Br}(X)/\text{Br}(\mathbb{R}) \neq 0$ is possible only in the last case.

Then the pencil of quadrics in \mathbf{P}^4 associated with X contains four rank-4 quadrics of negative discriminant. We may write each of them in the shape

$$-c_0 T_{i_0}^2 + c_1 T_{i_1}^2 + c_2 T_{i_2}^2 + c_3 T_{i_3}^2 = 0,$$

for $c_0, \dots, c_3 > 0$, and say that the variable T_{i_0} is distinguished by the form considered.

We claim that not all four forms may distinguish the same variable. Indeed, if that would be the case then we also had $-c'_0 T_{i_0}^2 + c'_1 T_{i_1}^2 + c'_2 T_{i_2}^2 + c'_4 T_{i_4}^2 = 0$, which shows that the form in the pencil that does not involve T_{i_0} has opposite signs at $T_{i_3}^2$ and $T_{i_4}^2$. The same argument for all combinations of two of the four quadratic forms enforces six opposite signs among the four coefficients of $T_{i_1}^2, \dots, T_{i_4}^2$, a contradiction.

Thus, X may be given by two equations of the form

$$\begin{aligned} -c_0 T_{i_0}^2 + c_1 T_{i_1}^2 + c_2 T_{i_2}^2 + c_3 T_{i_3}^2 &= 0, \\ -d_0 T_{j_0}^2 + d_1 T_{j_1}^2 + d_2 T_{j_2}^2 + d_3 T_{j_3}^2 &= 0, \end{aligned}$$

for $c_k, d_k > 0$, $i_0 \neq j_0$, and $\{i_0, \dots, i_3\} \cup \{j_0, \dots, j_3\} = \{0, \dots, 4\}$. These equations imply $x_{i_0} \neq 0$ and $x_{j_0} \neq 0$ for every real point $x \in X(\mathbb{R})$. In particular, $X(\mathbb{R})$ has at least two connected components, given by the two possible signs of x_{i_0}/x_{j_0} . Clearly, these two components are interchanged under the operation of $\text{Aut}'(X)$.

We finally note that a real degree four del Pezzo surface cannot have more than two connected components [Silh, Chapter III, Theorem 3.3]. \square

Remark 3.6. The stronger statement that if $X(\mathbb{R})$ splits into two connected components then the operation of $\text{Aut}'(X)$ interchanges them is true, as well.

Indeed, by blowing up a real point not lying on any exceptional curve, one obtains a real cubic surface that has two connected components. According to L. Schläfli [Sch, pp. 114f.], there are exactly five real types of real cubic surfaces and those correspond in modern language to the four conjugacy classes of order-2 subgroups in $W(E_6)$ together with the trivial group. Only for one of these five cases,

the Brauer group is nontrivial [Ja, Appendix, Table 2], it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ then, and that is the single case in which the surface is disconnected. We will not make use of this observation.

Proof of Theorem 1.3. Let X be a diagonalizable degree four del Pezzo surface over \mathbb{Q} that has an adelic point and a Brauer class $\alpha \in \text{Br}(X)$ working at the infinite place. We note that, since X has an adelic point, it clearly has a real point. The local evaluation $\text{ev}_{\alpha, \infty}(x)$ for $x \in X(\mathbb{R})$ is defined using the restriction homomorphism $x^*: \text{Br}(X) \rightarrow \text{Br}(\mathbb{R})$, which factors via $\text{Br}(X_{\mathbb{R}})$. Hence, non-constancy of $\text{ev}_{\alpha, \infty}$ implies that the restriction $\alpha_{\mathbb{R}} \in \text{Br}(X_{\mathbb{R}})/\text{Br}(\mathbb{R})$ is a nonzero class.

In this case, Lemma 3.5 shows that $X(\mathbb{R})$ splits into two connected components. Moreover, there exists an automorphism $\sigma \in \text{Aut}'(X_{\mathbb{R}}) = \text{Aut}'(X)$ interchanging these. Since $\text{ev}_{\alpha, \infty}$ is locally constant, this implies that σ is a witness for the non-constancy of the local evaluation map at ∞ . In other words, the natural homomorphism $i_{\alpha}: \text{Aut}'(X) \rightarrow \text{Br}(\mathbb{Q})$ has in its image a class $i_{\alpha}(\sigma)$ with a non-zero component at infinity.

According to global class field theory [Ta, Section 10, Theorem B], $i_{\alpha}(\sigma)$ necessarily has a nonzero component at a second place $\nu \neq \infty$. Consequently, α works at the place ν , too, which implies the claim. \square

4. SURFACES WITH A BRAUER CLASS WORKING AT A PRESCRIBED SET OF PLACES

The goal of this section is to prove Theorem 1.4. We distinguish between the cases $\#S > 1$, $\#S = 1$, and $S = \emptyset$. The family below will serve us in all cases.

4.1. A family of degree four del Pezzo surfaces. For $D, A_1, A_2, B \in \mathbb{Q}$, let $S := S^{(D; A_1, A_2, B)} \subset \mathbb{P}_{\mathbb{Q}}^4$ be given by the system of equations

$$\begin{aligned} (6) \quad & -A_1(T_0 - T_1)(T_0 + T_1) = T_3^2 - DT_4^2, \\ (7) \quad & -A_2(T_0 - T_2)(T_0 + T_2) = T_3^2 - B^2DT_4^2. \end{aligned}$$

Theorem 4.1. *Let D, A_1, A_2, B be nonzero rational numbers.*

A.a) *Then S is not a cone. The degree-5 scheme \mathcal{S}_X has a point at infinity and four others, which are the roots of a completely reducible polynomial of degree four having discriminant $\Delta := A_1^2(A_1 - A_2)^2(A_1B^2 - A_2)^2B^4(B - 1)^2(B + 1)^2/A_2^6B^{12}$.*

b) *S has the \mathbb{Q} -rational point $(1:1:1:0:0) \in X(\mathbb{Q})$.*

c) *If $\Delta \neq 0$ then the five rank-4 discriminants are, up to perfect square factors, given by D , D , as well as $DA_1A_2(A_1 - A_2)(B^2 - 1)$, $A_1A_2(A_1B^2 - A_2)(B^2 - 1)$, and $D(A_1 - A_2)(A_1B^2 - A_2)$.*

B.a) *There is a Brauer class $\alpha \in \text{Br}(X)$ extending that of the quaternion algebra $(\mathbb{Q}(\sqrt{D})(X), \tau, \frac{T_0+T_1}{T_0+T_2})$ over the function field $\mathbb{Q}(X)$.*

b) *Moreover, one has $\text{ev}_{\alpha, \nu}(x) = 0$ for $x = (1:1:1:0:0)$ and every $\nu \in \Omega$.*

c) *At an arbitrary place $\nu \in \Omega$, the local evaluation map $\text{ev}_{\alpha, \nu}$ is constant if one of the following conditions holds.*

- $\nu = p$ is a finite place, $p \neq 2$, and p divides neither D , nor A_1 , nor A_2 , nor B .
 - $\nu = p$ splits in $\mathbb{Q}(\sqrt{D})$, or $\nu = \infty$ and $D > 0$.
 - D is square-free, $\nu = p$ is a finite place, $p|D$, $p \neq 2$, $\gcd(B, D) = 1$, $(\frac{-A_1}{p}) = 1$, and $A_1 \equiv A_2 \pmod{p}$.
- d) At a place ν , the local evaluation map $\text{ev}_{\alpha, \nu}$ cannot be constant if $(-A_1, D)_\nu = -1$ or $(-A_2, D)_\nu = -1$.

Proof. A.a) and c) are standard calculations, while b) is directly checked. Moreover, B.a) is a direct application of Proposition 2.2.a) and the assertion of b) follows from the fact that $\frac{x_0+x_1}{x_0+x_2} = 1$ for $x = (1:1:1:0:0)$.

B.c) The sufficiency of the first condition is Criterion 2.6, while that of the second was shown in Proposition 2.2.b). In order to establish the sufficiency of the third, we argue as follows.

First of all, the prime p ramifies in $\mathbb{Q}(\sqrt{D})$. A p -adic unit $u \in \mathbb{Q}_p$ is a local norm from $\mathbb{Q}(\sqrt{D})$ if and only if $(u \bmod p) \in \mathbb{F}_p^*$ is a square. Moreover, we note that $(\frac{-A_1}{p}) = (\frac{-A_2}{p}) = 1$ implies that each of the four rational functions $\frac{T_0 \pm T_1}{T_0 \pm T_2}$ defines the Brauer class α .

Let now $x \in X(\mathbb{Q}_p)$ be any p -adic point. Normalize the coordinates x_0, \dots, x_4 such that each is a p -adic integer and at least one is a unit. If $p|x_0$ and $p|x_1$ or $p|x_0$ and $p|x_2$ then the equations imply that all coordinates must be divisible by p , a contradiction. Modulo p , we have $\bar{x}_0^2 - \bar{x}_1^2 = \bar{x}_0^2 - \bar{x}_2^2$, hence $\bar{x}_1 = \pm \bar{x}_2$, which implies that one of the four quotients $\frac{x_0 \pm x_1}{x_0 \pm x_2}$ is congruent to 1 modulo p , and therefore a norm.

B.d) We note first that X has $X(\mathbb{Q}_\nu)$ -rational points such that $x_0 \neq \pm x_1$ and $x_0 \neq x_2$. Indeed, putting $x_0 := 1$ and choosing x_3 and x_4 sufficiently close to 0 in the ν -adic topology, we see that (6) and (7) become soluble when viewed as equations for x_1 and x_2 , respectively.

Now, let us assume without loss of generality that $(-A_1, D)_\nu = -1$. Then the automorphism $\sigma: (T_0 : \dots : T_4) \mapsto (T_0 : (-T_1) : T_2 : T_3 : T_4)$ changes the rational function $\frac{T_0+T_1}{T_0+T_2}$ by a factor of

$$\frac{T_0-T_1}{T_0+T_1} = -\frac{1}{A_1} \frac{T_3^2 - DT_4^2}{(T_0+T_1)^2},$$

which is a ν -adic non-norm from $\mathbb{Q}(\sqrt{D})$ since $(-A_1, D)_\nu = -1$. This shows that $i_\alpha(\sigma)$ has a nonzero component at ν , i.e. that the automorphism σ witnesses the non-constancy of the local evaluation map $\text{ev}_{\alpha, \nu}$. \square

4.2. More than one place. Let $S \subset \Omega$ be a given set that consists of at least two places. We incorporate the notation $\{p_1, \dots, p_r\} = S \setminus \{2, \infty\}$.

To construct a diagonalizable degree four del Pezzo surface such that a nontrivial Brauer class works exactly at the places in S , we first choose a square-free integer $D \neq 0$ satisfying the following conditions.

- $D > 0$ if and only if $\infty \notin S$.
- $D \equiv 3 \pmod{4}$ in the case that $2 \in S$, and $D \equiv 1 \pmod{8}$ when $2 \notin S$.

• D is divisible by p_1, \dots, p_r and has exactly one further prime divisor, which we call q .

That such a choice of D is possible follows immediately from the fact that there are infinitely many primes in every odd residue class modulo 8.

Now write $S = S_1 \cup S_2$ as a union of two not necessarily disjoint subsets of even size. This is possible, because of $\#S \geq 2$. In addition, we may put 2 into both subsets in case it occurs as an element of S , and the same for ∞ .

Next, we choose primes $A_1 \neq A_2$ not dividing D such that, for $i = 1, 2$,

$$(8) \quad (-A_i, D)_\nu = -1 \iff \nu \in S_i.$$

To see that this may be achieved, we observe at first that $(-A_i, D)_\nu = 1$ for all places $\nu \neq 2, \infty; p_1, \dots, p_r, q$, and A_i . The requirement at $\nu = 2$ may be realized by choosing $A_i \equiv 1 \pmod{4}$, the condition at $\nu = \infty$ is implied by the choice that A_i is positive. Furthermore, we require

$$\left(\frac{-A_i}{p_j}\right) = \begin{cases} -1 & \text{if } p_j \in S_i, \\ 1 & \text{otherwise,} \end{cases}$$

and $\left(\frac{-A_i}{q}\right) = 1$. Let us impose, in addition, the condition that

$$(9) \quad A_2 \equiv A_1 \pmod{q}.$$

All these are congruence conditions modulo distinct odd primes. Therefore, the existence of a prime A_1 satisfying (8) for all places except, possibly, A_1 itself, is implied by Dirichlet's Theorem on primes in arithmetic progressions. Moreover, as $\#S_i$ is even, we have $(-A_i, D)_{A_i} = 1$ by the Hilbert reciprocity law [Ne, Chapter VI, Theorem 8.1].

In a completely analogous manner, Dirichlet's Theorem and the Hilbert reciprocity law imply the existence of a prime $A_2 \neq A_1$ fulfilling (8) and (9).

We may now formulate the main result of this paragraph.

Theorem 4.2. *Let the integers D , A_1 , and A_2 be chosen as above.*

a) *Then, for every integer $B \geq 2$, the surface $X \subset \mathbf{P}_{\mathbb{Q}}^4$ given by*

$$\begin{aligned} -A_1(T_0 - T_1)(T_0 + T_1) &= T_3^2 - DT_4^2, \\ -A_2(T_0 - T_2)(T_0 + T_2) &= T_3^2 - B^2DT_4^2 \end{aligned}$$

is nonsingular and has a \mathbb{Q} -rational point.

b) *There is a Brauer class $\alpha \in \text{Br}(X)$ extending that of the quaternion algebra $(\mathbb{Q}(\sqrt{D})(X), \tau, \frac{T_0+T_1}{T_0+T_2})$ over the function field $\mathbb{Q}(X)$.*

c) *The Brauer class α works at every place $\nu \in S$. If B is a prime number that splits in $\mathbb{Q}(\sqrt{D})$ then α does not work at any other place.*

d) *There are infinitely many prime numbers B splitting in $\mathbb{Q}(\sqrt{D})$ such that $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. a) follows from Theorem 4.1.A.a) and b). b) is Theorem 4.1.B.a).

c) Our choices of A_1 , A_2 , and D guarantee that Theorem 4.1.B.d) applies to every $\nu \in S$. On the other hand, as B is a prime that splits in $\mathbb{Q}(\sqrt{D})$, Theorem 4.1.B.c) shows constancy of the evaluation map $\text{ev}_{\alpha, \nu}$ for all other places.

d) According to Fact 2.3, in order to exclude the option $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, we have to choose the parameter B such that neither of the terms

$$A_1 A_2 (A_1 - A_2) (B^2 - 1), \quad D A_1 A_2 (A_1 B^2 - A_2) (B^2 - 1), \quad \text{and} \quad (A_1 - A_2) (A_1 B^2 - A_2)$$

is a perfect square. By Siegel's Theorem on integral points on elliptic curves [Silv, Theorem IX.4.3], the term in the middle is a square only finitely many times. The two others lead to Pell-like equations, the integral solutions of which are known to have exponential growth, cf. for example [Ch, Chapter XXXIII, §§15–18]. The assertion follows. \square

4.3. No place. It is not at all hard to write down a diagonalizable degree four del Pezzo surface X such that a nontrivial Brauer class works at no place at all.

Example 4.3. Let $X \subset \mathbf{P}_{\mathbb{Q}}^4$ be the surface given by

$$\begin{aligned} -(T_0 - T_1)(T_0 + T_1) &= T_3^2 - 17T_4^2, \\ -103(T_0 - T_2)(T_0 + T_2) &= T_3^2 - 68T_4^2. \end{aligned}$$

Then X is nonsingular and $X(\mathbb{Q}) \neq \emptyset$. Moreover, $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ but the nontrivial Brauer class works at no place.

Proof. The first two assertions follow from Theorem 4.1.A.b) and a). The discriminants of the five rank-4 forms are, up to square factors, 17, 17, 66, 206, and 3399 such that, by Facts 2.4.a) and 2.3, we have $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

Let $\alpha \in \text{Br}(X)$ be nontrivial. By Theorem 4.1.B.c), the local evaluation map is constant at all places $\nu \neq 2, 17, 103$, and ∞ . Moreover, it is constant at $\nu = \infty$ as the field $\mathbb{Q}(\sqrt{17})$ is real-quadratic. Constancy at $\nu = 2$ and 103 is clear, too, since these primes split in $\mathbb{Q}(\sqrt{17})$. Finally, $\text{ev}_{\alpha, 17}$ is constant as $(\frac{-1}{17}) = 1$ and $103 \equiv 1 \pmod{17}$. \square

4.4. Exactly one place. The examples here are necessarily a bit different, as the 16 automorphisms must not witness the non-constancy of the evaluation map. We may nonetheless work with the family from Theorem 4.1.

Example 4.4. Let l be a prime number such that $l \equiv 3 \pmod{4}$. Choose a prime $D \equiv 1 \pmod{8}$ such that $(\frac{D}{l}) = -1$ and another prime $A > l$ such that $A \equiv 1 \pmod{D}$ and $(A^2 - 1)(A^2 - l^2)$ is a non-square.

Then the surface $X \subset \mathbf{P}_{\mathbb{Q}}^4$ given by

$$\begin{aligned} -(T_0 - T_1)(T_0 + T_1) &= T_3^2 - DT_4^2, \\ -A^2(T_0 - T_2)(T_0 + T_2) &= T_3^2 - l^2 DT_4^2 \end{aligned}$$

is nonsingular and has a \mathbb{Q} -rational point. Moreover, $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial class works exactly at the place l .

Proof. We first note that the restrictions on D and A are easy to fulfill due to Dirichlet's and Siegel's theorems. Furthermore, the first three assertions follow directly from Theorem 4.1.A.a) and b), as well as Facts 2.4.a) and 2.3. The nontrivial Brauer class $\alpha \in \text{Br}(X)$ may be understood as an extension of the quaternion algebra $(\mathbb{Q}(\sqrt{D})(X), \tau, \frac{T_0+T_1}{T_0+T_2})$ over $\mathbb{Q}(X)$ to the whole scheme X . Moreover, any of the quotients $\frac{T_0 \pm T_1}{T_0 \pm T_2}$ defines the same Brauer class. Theorem 4.1.B.c) implies that the local evaluation map is constant at all places $\nu \neq l$.

Non-constancy of $\text{ev}_{\alpha,l}$: Note that l is an inert prime, since $(\frac{D}{l}) = -1$. An element $u \in \mathbb{Q}_l^*$ is a local norm from $\mathbb{Q}(\sqrt{D})$ if and only if $\nu_l(u)$ is even.

For $\underline{x} = (1:1:1:0:0)$, we have $\text{ev}_{\alpha,l}(\underline{x}) = 0$ by Theorem 4.1.B.b). On the other hand, the substitutions $T_0 = lT'_0$, $T_1 = T'_1$, $T_2 = lT'_2$, $T_3 = lT'_3$, and $T_4 = T'_4$ yield a different model X' of X that is given by

$$\begin{aligned} -(lT'_0 - T'_1)(lT'_0 + T'_1) &= l^2 T_3'^2 - DT_4'^2, \\ -A^2(T'_0 - T'_2)(T'_0 + T'_2) &= T_3'^2 - DT_4'^2. \end{aligned}$$

Moreover, $\frac{T_0+T_1}{T_0+T_2} = \frac{lT'_0+T'_1}{lT'_0+lT'_2} = \frac{1}{l} \frac{lT'_0+T'_1}{T'_0+T'_2}$. It suffices to find a \mathbb{Q}_l -rational point on X' so that $\frac{lT'_0+T'_1}{T'_0+T'_2}$ is a l -adic unit.

The reduction of X' modulo l is given by

$$\begin{aligned} T_1'^2 &= -\overline{D}T_4'^2, \\ -\overline{A}^2(T'_0 - T'_2)(T'_0 + T'_2) &= T_3'^2 - \overline{D}T_4'^2. \end{aligned}$$

We observe that the first equation has a nontrivial solution, as $(\frac{D}{l}) = -1$ and $l \equiv 3 \pmod{4}$ together imply that $(-\overline{D}) \in \mathbb{F}_l^*$ is a square. Let $\rho \in \mathbb{F}_l^*$ be one of its square roots.

The Jacobian matrix associated to a point $x \in X'(\mathbb{F}_l)$ is

$$\begin{pmatrix} 0 & 2x_1 & 0 & 0 & 2\overline{D}x_4 \\ -2\overline{A}^2x_0 & 0 & 2\overline{A}^2x_2 & -2x_3 & 2\overline{D}x_4 \end{pmatrix},$$

which shows that points such that $x_1 \neq 0$ are non-singular. For instance, there is the non-singular \mathbb{F}_l -rational point

$$x = (\frac{\overline{A}^2 + \overline{D}}{2\overline{A}^2} : \rho : \frac{\overline{A}^2 - \overline{D}}{2\overline{A}^2} : 0 : 1) \in X'(\mathbb{F}_l).$$

Here, $\frac{x_1}{x_0+x_2} = \rho$ is a nonzero element in \mathbb{F}_l . Hence, for every l -adic point that lifts x , the value of the quotient $\frac{lT'_0+T'_1}{T'_0+T'_2}$ is an l -adic unit as required. The assertion follows. \square

In order to provide the corresponding example in the $l \equiv 1 \pmod{4}$ case, we need the following lemma.

Lemma 4.5. *Let \mathbb{F}_l be a finite field of characteristic $\neq 2$. Then there exists an element $\sigma \in \mathbb{F}_l$ such that $2(1 + \sigma^2)$ is a non-square in \mathbb{F}_l .*

Proof. If $l = 3$ then put $\sigma := 0$. Otherwise, i.e. for $l \geq 5$, let $c \in \mathbb{F}_l$ be any non-square. The equation $cT_0^2 = 2(T_1^2 + T_2^2)$ defines a conic over \mathbb{F}_l , which has exactly $(l+1)$ \mathbb{F}_l -rational points. Among them, at most four have $x_1 = 0$ or $x_0 = 0$. For the others, $\sigma := x_2/x_1$ fulfills the required condition. \square

Example 4.6. Let l be a prime number that is either $l \equiv 1 \pmod{4}$ or $l = 2$. If $l = 2$ choose $B := 2$, otherwise let B be an odd prime number that is split in $\mathbb{Q}(\sqrt{l})$ and such that neither $l(l-1)(B^2-1)$, nor $(lB^2-1)(B^2-1)$, nor $(l-1)(lB^2-1)$ is a perfect square.

Then the surface $X \subset \mathbf{P}_{\mathbb{Q}}^4$ given by

$$(10) \quad -l(T_0 - T_1)(T_0 + T_1) = T_3^2 - lT_4^2,$$

$$(11) \quad -(T_0 - T_2)(T_0 + T_2) = T_3^2 - B^2lT_4^2$$

is nonsingular and has a \mathbb{Q} -rational point. Moreover, $\text{Br}(X)/\text{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial class works exactly at the place l .

Proof. Once again, the restrictions on B are easy to fulfill due to Dirichlet's and Siegel's theorems. Furthermore, the first three assertions follow directly from Theorem 4.1.A.a) and b) as well as Facts 2.4.a) and Fact 2.3. There is a Brauer class $\alpha \in \text{Br}(X)$, which may be understood as an extension of the quaternion algebra $(\mathbb{Q}(\sqrt{l})(X), \tau, \frac{T_0+T_1}{T_0+T_2})$ over $\mathbb{Q}(X)$ to the whole scheme X . Moreover, any of the quotients $\frac{T_0 \pm T_1}{T_0 \pm T_2}$ defines the same Brauer class. Finally, Theorem 4.1.B.c) implies that the local evaluation map is constant at all places $\nu \neq 2, l$. Thus, all which remains to be shown is that $\text{ev}_{\alpha, l}$ is non-constant and that $\text{ev}_{\alpha, 2}$ is constant in the case $l \equiv 1 \pmod{4}$.

Non-constancy of $\text{ev}_{\alpha, 2}$ for $l = 2$: For $\mathbb{Q}(\sqrt{2})$, the prime $p = 2$ is ramified. A 2-adic unit u is a local norm from $\mathbb{Q}(\sqrt{2})$ if and only if $u \equiv \pm 1 \pmod{8}$.

For $\underline{x} = (1 : 1 : 1 : 0 : 0)$, we have $\text{ev}_{\alpha, 2}(\underline{x}) = 0$ by Theorem 4.1.B.b). On the other hand, there is the 2-adic point $x = (1 : 0 : \sqrt{-7} : 0 : 1) \in X(\mathbb{Q}_2)$. Observe that $(-7) \equiv 1 \pmod{8}$ implies that (-7) is a square in \mathbb{Q}_2 . Moreover, we may choose $\sqrt{-7} \in 5 + 16\mathbb{Z}_2$ since $5^2 \equiv -7 \pmod{32}$. Then $\frac{x_0+x_1}{x_0+x_2} = 1/(1 + \sqrt{-7})$, which is in the residue class $\frac{1}{2} \cdot (3 \pmod{8})$. Consequently, $\text{ev}_{\alpha, 2}(x) = \frac{1}{2}$.

Non-constancy of $\text{ev}_{\alpha, l}$ for $l \equiv 1 \pmod{4}$: For $\mathbb{Q}(\sqrt{l})$, the prime l is ramified. A l -adic unit u is a local norm from $\mathbb{Q}(\sqrt{l})$ if and only if $(u \pmod{l}) \in \mathbb{F}_l^*$ is a square.

For $\underline{x} = (1 : 1 : 1 : 0 : 0)$, we have $\text{ev}_{\alpha, l}(\underline{x}) = 0$. On the other hand, the substitution $T_3 = lT'_3$ yields a different model X' of X that is given by

$$\begin{aligned} (T_0 - T_1)(T_0 + T_1) &= T_4^2 - lT_3'^2, \\ (T_0 - T_2)(T_0 + T_2) &= l(B^2T_4^2 - lT_3'^2). \end{aligned}$$

The reduction of X' modulo l is given by

$$\begin{aligned} (T_0 - T_1)(T_0 + T_1) &= T_4^2, \\ (T_0 - T_2)(T_0 + T_2) &= 0. \end{aligned}$$

From this, we see that the Jacobian matrix associated to a point $x \in X'(\mathbb{F}_l)$ is

$$\begin{pmatrix} 2x_0 - 2x_1 & 0 & 0 & -2x_4 \\ 2x_0 & 0 & -2x_2 & 0 & 0 \end{pmatrix},$$

which shows that points such that $x_0 = x_2 \neq 0$ are non-singular. For instance, for any $\sigma \in \mathbb{F}_l$ such that $\sigma^2 \neq -1$, there is the non-singular \mathbb{F}_l -rational point

$$x = ((1 + \sigma^2):2\sigma:(1 + \sigma^2):0:(1 - \sigma^2)) \in X(\mathbb{F}_l).$$

If, moreover, σ is chosen as in Lemma 4.5 then $\frac{x_0+x_1}{x_0+x_2} = \frac{(1+\sigma)^2}{2(1+\sigma^2)}$, which is a non-square. Then, for every l -adic point that lifts x , the local evaluation map has value $\frac{1}{2}$.

Constancy of $\text{ev}_{\alpha,2}$ for $l \equiv 1 \pmod{4}$: If $l \equiv 1 \pmod{8}$ then the prime $p = 2$ is split in $\mathbb{Q}(\sqrt{l})$ and there is nothing to prove.

On the other hand, assume that $l \equiv 5 \pmod{8}$, in which case 2 is an inert prime. Then an element $u \in \mathbb{Q}_2^*$ is a local norm from $\mathbb{Q}(\sqrt{l})$ if and only if $\nu_2(u)$ is even. Moreover, any 2-adic point $x \in X(\mathbb{Q}_2)$ may be represented by coordinates x_0, \dots, x_4 that are 2-adic integers, at least one of which is a unit. It is now a routine matter to determine all quintuples of residues modulo 8, one of which is odd, that satisfy the system (10,11) of equations modulo 8. From the list obtained, one already sees that $\text{ev}_{\alpha,2}(x) = 0$ in each case. We leave the details to the reader. \square

5. A CONSEQUENCE CONCERNING DEL PEZZO SURFACES OF LOW DEGREE

The aim of this section is to prove Theorem 1.5. For this, we need some technical facts about point sets on \mathbf{P}^2 that lie in general position, which should be well-known among experts. We decided to include them due to the lack of a suitable reference.

Let K be an algebraically closed field. One says that $r \leq 8$ distinct points $P_1, \dots, P_r \in \mathbf{P}^2(K)$ lie *in general position*, if no three of them lie on a line, no six on a conic, and no eight on a cubic that has a singular point at one of them. It is well known, cf. [Do, Proposition 8.1.25], [Is, Section 3], or [De, Theorem 1], that the blow-up of \mathbf{P}_K^2 in P_1, \dots, P_r is a del Pezzo surface if and only if these points lie in general position.

Lemma 5.1. *Let K be an algebraically closed field and $P_1, \dots, P_7 \in \mathbf{P}^2(K)$ be seven points in general position. Then there exists a nonsingular cubic through all of them.*

Proof. By [Ha, Corollary V.4.4.a) and Proposition V.4.3], the linear system of all cubics through P_1, \dots, P_7 is two-dimensional and has no unassigned base points. Blowing up P_1, \dots, P_7 , we find a two-dimensional linear system on a non-singular surface that is base point free.

A version of Bertini's theorem [Ha, Corollary III.10.9] shows that the generic element of this linear system is smooth. Projecting down to \mathbf{P}^2 , we see that no singular points may occur, except for P_1, \dots, P_7 .

However, according to [Man, Theorem 26.3], there are only seven cubics through P_1, \dots, P_7 that are singular at one of them. Therefore, the generic member must be nonsingular, as required. \square

Proposition 5.2. *Let K be an algebraically closed field and $P_1, \dots, P_5 \in \mathbf{P}^2(K)$ be five points in general position. Then there exist $P_6, P_7, P_8 \in \mathbf{P}^2(K)$ such that $P_1, \dots, P_8 \in \mathbf{P}^2(K)$ lie in general position.*

Proof. First choose $P_6 \in \mathbf{P}^2(K)$ not lying on any of the 10 lines through two of the five points, and neither on the conic through all of them. Next, choose $P_7 \in \mathbf{P}^2(K)$ outside the 15 lines through two of the six points and outside the six conics through five of them. All these requirements exclude only a one-dimensional subset of \mathbf{P}^2 .

When we finally choose P_8 , we have to be slightly more careful, as singular cubics need to be taken into consideration. Clearly, we have to choose $P_8 \in \mathbf{P}^2(K)$ outside the 21 lines through two of the seven points, outside the 21 conics through five of them, and outside the seven cubics through the points P_1, \dots, P_7 that are singular at one of them. This excludes, once again, only a one-dimensional subset of \mathbf{P}^2 .

There is one final condition. P_8 must not be the singular point of a cubic through P_1, \dots, P_7 . To analyze this requirement, recall [Ha, Corollary V.4.4.a)] that the linear system of all cubics through these seven points is two dimensional. By Lemma 5.1, it contains a nonsingular curve. Thus, singular curves form an at most one-dimensional subfamily. Moreover, none of these cubics may have multiple components, so that the total set of singular points occurring is at most one-dimensional. This completes the proof. \square

Corollary 5.3. *Let K be an algebraically closed field and X be a del Pezzo surface of degree four over K . Then the set of all $(P, Q, R) \in X^3(K)$ such that $\mathrm{Bl}_{\{P, Q, R\}}(X)$ is a del Pezzo surface of degree one is Zariski open and non-empty.*

Proof. X is isomorphic to \mathbf{P}^2 , blown up in five points in general position. Proposition 5.2 shows that the set considered is non-empty. Moreover, being in general position is an open condition, so that Zariski openness is clear. \square

Proof of Theorem 1.5. For $d = 4$, this is Theorem 1.1. It yields a degree four del Pezzo surface X having a \mathbb{Q} -rational point such that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial Brauer class works exactly at the places in S . We note that, by [SSk, Theorem 0.1], the \mathbb{Q} -rational points on X are Zariski dense.

Moreover, Brauer groups do not change under blowup and the local evaluation maps are compatible in the sense that $\mathrm{ev}_{\alpha, p}(\pi(x)) = \mathrm{ev}_{\pi^* \alpha, p}(x)$. Thus, given some integer $1 \leq d < 4$, let us blow up X in $(4 - d)$ \mathbb{Q} -rational points. This clearly yields a surface Y over \mathbb{Q} that has a \mathbb{Q} -rational point and fulfills the conditions that $\mathrm{Br}(Y)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial Brauer class works exactly at the places in S .

As $K_Y^2 = 4 - (4 - d) = d$, it only remains to ensure that we may choose the blow-up points in such a way that Y becomes a del Pezzo surface. In view of [Man, Corollary 24.5.2.i)], it suffices to do this in the case when $d = 1$.

For this, let us fix an algebraic closure $\overline{\mathbb{Q}}$ and view \mathbb{Q} as a subfield of it. By Corollary 5.3, we know that the set of all $(P, Q, R) \in X^3(\overline{\mathbb{Q}})$ that yield a del Pezzo surface is Zariski open and non-empty. On the other hand, since $X(\mathbb{Q})$ is Zariski dense in $X(\overline{\mathbb{Q}})$, we also have that $X^3(\mathbb{Q})$ is Zariski dense in $X^3(\overline{\mathbb{Q}})$. As a non-empty open subset and a dense one necessarily have a point in common, the proof is complete. \square

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